WAVE PROPAGATION IN A WEAKLY DISPERSIVE MEDIUM

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Information on the nature of a process occurring at a certain distance from a receiver is transmitted by waves in the medium by the source. As the signal propagates it is attenuated and distorted. Several papers have considered the factors which lead to a significant transformation of the signal [1-3].

In a systematic approach to the problem, not only the dissipative properties of the medium must be taken into account, but also its dispersive properties. Indeed, when there is partial absorption of the wave by the medium, the medium obtains momentum and this leads to motion, which in turn changes the local velocity of the signal and therefore is equivalent to dispersion. In general, the relation between the wave parameters of the medium is nonlocal: Processes with different time and spatial scales have their own time and spatial parameters which determine the dependence of the wave number k on frequency p and the real and imaginary parts of the wave number $k(p) = k'(p) + ik''(p)$ are connected by a fundamental relation which follows from the principle of causality [1].

In order to find the explicit form of the functional relation between the wave parameters, the internal structure of the medium must be taken into account using a mathematical apparatus which can adequately describe the scaling phenomena determining the behavior of the medium for the range of the parameters under consideration.

A series of important results on the nature of the propagation of excitations in a relaxing medium were obtained by Mandel'shtam and Leontovich [2, 3]: The wave perturbation causes a loss of thermodynamic equilibrium in the medium and the medium returns to a different equilibrium state. For a perturbation which changes in time significantly more rapidly than the medium can readjust to the new state, the perturbation is propagated in the medium with the velocity c_∞; for changes of the external parameters which are slow compared to the relaxation time the propagation velocity of the perturbation is c_0.

In the present paper we note that a great deal of data [4] has shown that in geological media there is a weak dispersion waves over a wide range of wavelength. This fact can be used to obtain an expression for the mass velocity in the wave in a form convenient for analysis.

We solve the one-dimensional wave equation in a uniform isotropic medium in which the relation between the stress σ_{ij} and the deformation rate v_{ij} is given by an equation of the hereditary type

$$\sigma_{ij}(x, t) = \delta_{ij}\int_0^t K(t-t')\, v_{kk}(x, t')\, dt' + 2\int_0^t dt'\mu(t-t')\left[v_{ij}(x, t') - \frac{\delta_{ij}}{3} v_{kk}(x, t')\right],$$
$$v_{ij} = \frac{1}{2}\left[\frac{\partial}{\partial x_j} v_i + \frac{\partial}{\partial x_i} v_j\right], \qquad (1)$$

where K(t) and $\mu(t)$ are the time-dependent bulk modulus and shear modulus of the medium, respectively.

In the plane case of interest to us, (1) has the form

$$\sigma_{xx}(x, t) = \int_0^t E_\tau(t-t')\, v_{xx}(x, t')\, dt', \quad E_\tau(t) = K(t) + \frac{4}{3}\mu(t).$$

We assume that the relaxation properties of the medium can be described by a single characteristic parameter τ (the relaxation time) so that the dependence $E_\tau(t)$ on t and τ is in the form

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$$E_\tau(t) = E_0 + E_p \exp(-t/\tau).$$

Using the relation between $\sigma_{xx}(x, t)$ and $v_x(x, t)$, and also the equation of motion and the equation of continuity

$$\rho \frac{\partial}{\partial t} v_x = \frac{\partial}{\partial x} \sigma_{xx}, \quad \frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho v_x) = 0,$$

we obtain assuming that $E_0 \gg E_p$ the following expression for the mass velocity:

$$v_x(x, t) = \frac{1}{2\pi i} \int\limits_{-i\infty+a}^{+i\infty+a} dp \Phi_p(p) \exp\left[pt - \frac{x}{c_\infty} p \left(1 + \frac{1-\alpha}{2} \frac{1}{1+p\tau}\right)\right],$$
$$\Phi_p(p) = \int\limits_0^\infty v_x(0, t) \exp(-pt)\, dt, \quad \alpha = \frac{c_0^2}{c_\infty^2}, \quad c_\infty^2 = \frac{E_0 + E_p}{\rho}, \quad c_0^2 = \frac{E_0}{\rho}. \tag{2}$$

Evaluating the integral in (2), we have

$$v_x(x, t) = \Theta(t') \left\{ v_x(0, t') + \Delta^{1/2} \int\limits_0^{t'} d\xi v_x(0, \xi) I_1 (2\sqrt{\Delta(t'-\xi)}) (t'-\xi)^{-1/2} \exp[-\alpha(t'-\xi)/\tau] \right\} \exp(-\tau\Delta/\alpha),$$
$$\Delta = \alpha(1-\alpha) x / 2c_\infty \tau^2, \quad t' = t - \frac{x}{c_\infty}, \quad \Theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases} \tag{3}$$

If there is a sudden pulse $v_x(0, t) = v_0\delta(t)$ at $x = 0$ and at the initial instant of time, then at distances x and time t the signal takes the form

$$v_x(x, t) = v_0 \Theta(t') \left\{ \exp(-\tau\Delta/\alpha)\, \delta(t') + \Delta^{1/2} \frac{I_1(2\sqrt{\Delta t'})}{t'^{1/2} \exp\left[\frac{\tau}{\alpha}(\Delta + \alpha^2\tau^{-2} t')\right]} \right\}. \tag{4}$$

As the signal propagates in a weakly dispersive medium, its shape is distorted: The leading edge, moving with the velocity c_∞, decreases exponentially with distance (the maximum amplitude falls off the distance as $x^{-1/2}$, and the width of the pulse increases with distance as $x^{1/2}$). Propagation of a delta-function pulse of this kind was described in [5], where results were obtained for wave propagation when the dependence of the attenuation coefficient on frequency was linear or quadratic [in the case of the quadratic dependence the dispersion of the wave is infinite, $c_0 < p/k(p) < \infty$]. This frequency dependence of the attenuation coefficient is correct only in the long-wavelength approximation, and use of this relation for all frequencies would lead to a violation of the principle of causality.

When the parameters of the problem are such that $4\Delta t' \gg 1$, Eq. (4) takes the form

$$v_x(x, t) = \alpha v_0 \sqrt{\frac{c_\infty}{2\pi\tau x}} \exp[-\alpha\tau^{-1}(\tau\alpha^{-1}\sqrt{\Delta} - \sqrt{t'})^2].$$

When the time dependence of the source is of the form $v_x(0, t) = v_0\Theta(t)$, the calculation can be done as follows. We introduce the substitution $t' - \xi = u$ under the integral sign, differentiate $v_x(x, t)$ with respect to time, and then expand the resulting expression around the value $t' = x(1-\alpha)/2c_\infty$, which gives the largest contribution to the integral. We then integrate with respect to time and obtain the expression

$$v_x(x, t) = \frac{1}{2} v_0 \left[\Phi\left(\frac{t - \frac{x}{c_0}}{\tau\sqrt{\tau\Delta}}\right) + 1\right], \quad \Phi(z) = 2\pi^{-1/2} \int\limits_0^z \exp[-v^2]\, dv,$$

given in [6], where it was noted that there is a jump on the wave front in a dispersive medium. However the explicit form of the dependence of the velocity on position and time was not obtained and therefore the flow of the medium behind the front could not be studied.

In the general case of an arbitrary dependence of the input signal on time, the integral in (3) can be evaluated approximately using the fact that the function under the integral sign is rapidly varying. Except for a small neighborhood $(t' - \xi)\Delta \lesssim 1$ the integral in (3) can be written in the form

$$\Delta^{1/2} \int\limits_0^{t'} d\xi v_x(0, \xi) \frac{\exp[-\alpha\tau^{-1}(t'-\xi) - \tau\Delta/\alpha + 2\sqrt{\Delta(t'-\xi)}]}{\{4\pi[\Delta^{1/2}(t'-\xi)]^{1/2}\}^{1/2}}.$$

Near the point $\xi_m = t - x/c_0$ the function under the integral sign has a maximum. Expanding out the expression in the exponential up to quadratic terms near the extremum point we obtain

$$v_x(x, t) \cong \frac{\alpha}{\sqrt{4\pi\tau^3\Delta}} \int_0^{t'} d\xi v_x(0, \xi) \exp\left[-\frac{\alpha c_\infty}{2\tau(1-\alpha)x}(\xi_m - \xi)^2\right] + v_x(0, t') \exp[-\tau\Delta/\alpha]. \quad (5)$$

At large distances from the source, when $2c_\infty\tau \ll (1-\alpha)x \ll \alpha T_s^2 c_p/2\tau$, the dominant contribution to $v_x(x, t)$ is the extremum point ξ_m. In this case

$$v_x(x, t) \cong v_x(0, t - x/c_0).$$

If the distance from the source is such that $4c_\infty\tau^2(1-\alpha) \sim 2x\tau(1-\alpha) \ll \alpha c_\infty T_s^2$, then $v_x(x, t \cong v_x(0, t - x/c_0) + v_x(0, t - x/c_\infty) \exp\left[-\frac{(1-\alpha)x}{2c_\infty\tau}\right]$. When the characteristic time constant of the input signal T_s satisfies the inequality $\tau^2 \ll T_s^2 \ll 2(1-\alpha)\tau x/c_\infty\alpha$ then the exponential can be evaluated at $\xi = 0$ and taken outside of the integral sign and we get

$$v_x(x, t) \cong \frac{\alpha}{\sqrt{4\pi\Delta\tau^3}} \exp\left[-\frac{c_\infty}{2(1-\alpha)\tau x}\right] \int_0^\infty v_x(0, t')\, dt',$$

so that at large enough distances the excitation is independent of its original shape and will have the shape of a Gaussian, dying out with distance as $x^{-1/2}$.

A more detailed picture of the propagation of the signal can be obtained by numerical calculations.

If the input signals is such that $\int_0^\infty v_x(0, t)\, dt$ is small, then we can expand the integrand in (5) up to terms of order ξ_m/x. An expression for the mass velocity at large distances is then given by

$$v_x(x, t) \cong \frac{\alpha \exp\left[-\frac{c_\infty \xi_m^2}{2(1-\alpha)\tau x}\right]}{\sqrt{4\pi\Delta\tau^3}} \left\{\int_0^\infty v_x(0, t)\, dt + \frac{\alpha\xi_m c_\infty}{(1-\alpha)\tau x} \int_0^\infty dt\, t v_x(0, t)\right\},$$

which shows the oscillatory dependence of the velocity $v_x(x, t)$ on time at a fixed distance and also shows that the wave damps out with distance according to the dependence $\tilde{\alpha}x^{-1/2} + \tilde{\beta}x^{-3/2}$. This was apparently observed experimentally [7], where the negative phase of the signal was noted and also the dependence of the maximum amplitude on distance as $v_m \sim x^{-1.23}$. For certain $\tilde{\alpha}$ and $\tilde{\beta}$ such a dependence follows from (5) for a limited range of x.

When the input signal depends on time as $v_x(0, t) = v_0 \operatorname{Re} \exp[-\gamma t + i\omega t]$, the expression for the mass velocity takes the form

$$v_x(x, t) = \frac{v_0}{2} \operatorname{Re}\{\Phi(t'/\bar{\Delta} - \bar{\xi}) + \Phi(\bar{\xi})\} \exp[-\xi_m^2/\bar{\Delta}^2 + \bar{\xi}^2],$$

$$\bar{\Delta}^2 = (1-\alpha)\tau x/2c_\infty, \quad \bar{\xi} = \xi_m/\bar{\Delta} - \bar{\Delta}\gamma/2 + i\omega\bar{\Delta}/2.$$

Using the asymptotic form of $\Phi(x + iy)$ at large values of the real part of the argument, we obtain

$$v_x(x, t) \cong \frac{v_0}{2} \operatorname{Re}\{1 + \Phi(\bar{\xi})\} \exp[\bar{\xi}^2 - \xi_m^2/\bar{\Delta}^2]. \quad (6)$$

For an unattenuated signal $\gamma = 0$ and it follows from (6) that a harmonic wave dies out exponentially with distance: $v_x(x, t) \sim \exp[-\omega^2\tau x/c_\infty]$.

The analysis given here shows that the experimentally determined quantity $v_x(x, t)$ depends mainly on the relation between parameters of the medium such as the relaxation time τ, the dispersion $1 - \alpha$, the velocity of the signal, and the time duration of the source, and also the distance at which the signal is measured. In order to interpret the experimental results correctly, it is necessary to use the expression for $v_x(x, t)$ which corresponds to the parameters of the problem. Since some of the parameters are determined only by the properties of the medium (the wave velocity, relaxation time, dispersion) and the time duration of the source is determined by the experimental design, the analysis given here indicates a lack of similarity between small and large-scale experiments in a dispersive and dissipative medium.

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MEASUREMENT OF THE VELOCITY OF WEAK DISTURBANCES OF BULK DENSITY IN POROUS MEDIA

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Fire stops with fillings consisting of granulated materials are widely used in the chemical, gas, and petroleum industries [1]. Present semiempirical estimates make it possible in each given case to select the fillings necessary to protect against explosion during production. However, such estimates do not reveal the mechanism of interaction of combustion waves and, especially, detonation waves with bulk systems within a broad range of materials and sizes of the granules. Nevertheless, it was shown in [2-4] that the combustion and detonation of gases in inert porous media are quite different from combustion and detonation in the absence of a solid phase. This is manifest, for example, in anomalous combustion and detonation velocities.

To determine the mechanism responsible for these phenomena, it is important to consider the gasdynamic characteristics of bulk and porous systems, particularly the speed of sound. However, it must be noted that one feature of flows of two-phase gas–particle media is the presence of friction and heat exchange between the phases. This precludes the existence of nonsteady isentropic motions in such systems.

It is known [5] that the state of a gas in viscous flow can be described by the polytropy equation

$$p\rho^{-n} = \text{const},$$

where p and ρ are the pressure and density of the gas; n is the polytropy index. From here we introduce the notion of a characteristic velocity of the given process [5]:

$$u^2 = np\rho^{-1}. \quad (1)$$

When n = γ (γ is the ratio of the specific heats of the gas), i.e., in the case of isentropic flow, the characteristic velocity will be the speed of sound.

A large number of studies has been devoted to the question of the speed of sound (i.e., the rate of propagation of small pressure perturbations with constant entropy) in a gas–particle medium. These studies can be divided into two groups. The first group contains studies of systems with a slow volume concentration of solid (or liquid) particles. In this case, a gas suspension is equivalent to a gas with the speed of sound [6]

$$a_e^2 = a^2\Gamma\,[\gamma\,(1+\eta)\,(1-\beta)^2]^{-1}, \quad (2)$$

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